On the Use of the Coulomb Gauge in Solving Source-Excited Boundary Value Problems of Electromagnetics

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Abstract **—The advantages and difficulties associated with the use of the Coulomb gauge in solving source-excited boundary value problems of electromagnetics are examined. The correct dyadic Green's function for the Coulomb vector potential in a rectangular waveguide is derived to elucidate the discussion. A flaw in the usage of the Coulomb gauge in Smythe's *Static and Dynamic Electricity* is uncovered.**

I. Introduction

The SOLUTION of boundary value problems of electromagnetics is often facilitated by the introduction of the scalar and vector potentials, which are related by the so-called gauge condition [1]. These potentials are not unique and they depend on the gauge employed, the Lorentz gauge being the most common choice. Occasionally, the Coulomb gauge is used, usually when there are no free charges [1]. The main purpose of this paper is to discuss the advantages and difficulties associated with the use of the Coulomb gauge in solving source-excited boundary value problems of electromagnetics. To better illustrate the ideas, we have selected for detailed analysis the familiar rectangular waveguide geometry.

Smythe's classical textbook [2] is the only reference known to the authors in which the Coulomb gauge is employed to solve problems involving arbitrarily oriented, time-harmonic dipoles in waveguides and cavities. However, Smythe's analysis contains a subtle flaw, which we presume is not widely known to the electromagnetics community. This flaw, its origins and ways to remedy it are also addressed in this paper.

After the preliminaries of Section II, we derive in Section III the Coulomb dyadic Green's function for the rectangular waveguide by the eigenfunction expansion method [3], [4], which enables us to identify terms that are missing in the corresponding expressions given by Smythe [2]. In Section IV, we summarize Smythe's approach and point out its flaw. We draw conclusions and make recommendations in Section V.

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II. Preliminaries

The problem of interest is that of finding the electromagnetic field due to a time-harmonic ($e^{j\omega t}$ time convention) dipole in a homogeneous medium characterized by permittivity ϵ and permeability μ and enclosed, at least partially, by a perfectly conducting surface S having a unit normal vector $\hat{\boldsymbol{n}}$.

We introduce the magnetic vector potential $\boldsymbol{A}$ in the usual manner by relating it to the magnetic field as[1]

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} = \frac{1}{\mu}\nabla \times \boldsymbol{A}^s. \tag{1}$$

It then follows from Maxwell's equations that under the Coulomb condition, $\nabla \cdot \boldsymbol{A}^l = 0$, the vector potential satisfies the Helmholtz equation [1]:

$$(\nabla^2 + k^2)\boldsymbol{A}^s = -\mu \boldsymbol{J}^s \tag{2}$$

where $k^2 = \omega^2\mu\epsilon$, subject to the boundary condition $\hat{\boldsymbol{n}} \times \boldsymbol{A}^s = \boldsymbol{0}$ on S and (when S extends to infinity) the radiation condition. Hence, the Coulomb vector potential depends exclusively on the solenoidal part of the current density $\boldsymbol{J}$.

The electric field in the Coulomb gauge is given as

$$\boldsymbol{E} = -j\omega \boldsymbol{A}^s - \nabla\Phi = \boldsymbol{E}^s + \boldsymbol{E}^l \tag{3}$$

where the scalar potential Φ satisfies the Poisson equation:

$$\nabla^2\Phi = \frac{1}{j\omega\epsilon}\nabla \cdot \boldsymbol{J}^l \tag{4}$$

subject to the condition that $\Phi = 0$ on S and the require-

[1]The solenoidal (divergenceless) and lamellar (irrotational) parts of vectors and dyadics are denoted by, respectively, superscripts s and l. In the literature, the terms "transverse" and "longitudinal" are also used interchangeably with, respectively, "solenoidal" and "lamellar." This nomenclature is not followed here to avoid confusion with common waveguide terminology.

ment that Φ vanish at infinity. Hence, the Coulomb scalar potential depends exclusively on the lamellar part of $\boldsymbol{J}$.

Since Φ and thus $\boldsymbol{E}^l$ can be found relatively easily (this is illustrated for the rectangular waveguide in the Appendix), we focus here on the determination of $\boldsymbol{A}^s$, and thus $\boldsymbol{E}^s$ (cf. (3)). To facilitate this, we introduce the dyadic Green's function $\underline{\boldsymbol{G}}_A^s$, which satisfies

$$(\nabla^2 + k^2)\underline{\boldsymbol{G}}_A^s(\boldsymbol{r}|\boldsymbol{r}') = -\underline{\boldsymbol{\delta}}^s(\boldsymbol{r}-\boldsymbol{r}') \tag{5}$$

subject to the condition $\hat{\boldsymbol{n}} \times \underline{\boldsymbol{G}}_A^s = 0$ on S and the radiation condition at infinity. In the above, $\underline{\boldsymbol{\delta}}^s$ denotes the solenoidal part of the dyadic delta function $\underline{\boldsymbol{\delta}}(\boldsymbol{r}-\boldsymbol{r}') = \underline{\boldsymbol{I}}\delta(\boldsymbol{r}-\boldsymbol{r}') = \underline{\boldsymbol{\delta}}^s(\boldsymbol{r}-\boldsymbol{r}') + \underline{\boldsymbol{\delta}}^l(\boldsymbol{r}-\boldsymbol{r}')$, where $\underline{\boldsymbol{I}}$ is the idemfactor [3], [5]. Since $\underline{\boldsymbol{G}}_A^s$ is solenoidal, the operator ∇^2 in (5) can be replaced by $-\nabla \times \nabla \times$, if desired.

We recall [6] that the Green's dyadic for the Lorentz vector potential $\boldsymbol{A}$ satisfies

$$(\nabla^2 + k^2)\underline{\boldsymbol{G}}_A(\boldsymbol{r}|\boldsymbol{r}') = -\underline{\boldsymbol{\delta}}(\boldsymbol{r}-\boldsymbol{r}')$$
$$\hat{\boldsymbol{n}} \times \underline{\boldsymbol{G}}_A = 0 \qquad \nabla\cdot\underline{\boldsymbol{G}}_A = 0 \text{ on } S \tag{6}$$

whereas the Green's dyadic for the electric field is given as

$$(\nabla \times \nabla \times - k^2)\underline{\boldsymbol{G}}_E(\boldsymbol{r}|\boldsymbol{r}') = \underline{\boldsymbol{\delta}}(\boldsymbol{r}-\boldsymbol{r}')$$
$$\hat{\boldsymbol{n}} \times \underline{\boldsymbol{G}}_E = 0 \text{ on } S \tag{7}$$

so that

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu \int_V \underline{\boldsymbol{G}}_E(\boldsymbol{r}|\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')\,d\boldsymbol{r}' \tag{8}$$

where the integration is over the current-carrying volume V. We observe from the above that the solenoidal parts of $\underline{\boldsymbol{G}}_A$ and $\underline{\boldsymbol{G}}_E$ are equal, i.e., $\underline{\boldsymbol{G}}_A^s = \underline{\boldsymbol{G}}_E^s$. Consequently,

$$\underline{\boldsymbol{G}}_A^s(\boldsymbol{r}|\boldsymbol{r}') = \int_V \underline{\boldsymbol{\delta}}^s(\boldsymbol{r}-\boldsymbol{r}'')\cdot\underline{\boldsymbol{G}}_A(\boldsymbol{r}''|\boldsymbol{r}')\,d\boldsymbol{r}''$$
$$= \int_V \underline{\boldsymbol{\delta}}^s(\boldsymbol{r}-\boldsymbol{r}'')\cdot\underline{\boldsymbol{G}}_E(\boldsymbol{r}''|\boldsymbol{r}')\,d\boldsymbol{r}''. \tag{9}$$

We also note that

$$\underline{\boldsymbol{G}}_E^l(\boldsymbol{r}|\boldsymbol{r}') = -\frac{1}{k^2}\underline{\boldsymbol{\delta}}^l(\boldsymbol{r}-\boldsymbol{r}'). \tag{10}$$

III. Derivation of $\underline{\boldsymbol{G}}_A^s$ for the Rectangular Waveguide by Means of the Eigenfunction Expansion

Consider a perfectly conducting rectangular waveguide aligned along the z axis and with dimensions a and b along the x and y axes, respectively. To find $\underline{\boldsymbol{G}}_A^s$ for this geometry, we follow the Ohm–Rayleigh method, as described by Tai [4]. First, we expand $\underline{\boldsymbol{\delta}}^s$ in (5) as follows:

$$\underline{\boldsymbol{\delta}}^s(\boldsymbol{r}|\boldsymbol{r}') = \int_{-\infty}^{\infty} dh \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{2-\delta_0}{\pi a b k_c^2} \cdot\left[\boldsymbol{M}_{emn}(h)\boldsymbol{M}'_{emn}(-h) + \frac{1}{\kappa^2}\boldsymbol{N}_{omn}(h)\boldsymbol{N}'_{omn}(-h)\right] \tag{11}$$

where $\boldsymbol{M}_{emn}$ and $\boldsymbol{N}_{omn}$ are, respectively, the even and odd solenoidal eigenfunctions of the operator $\nabla \times \nabla \times$, satisfying the Dirichlet boundary conditions at the waveguide walls. These vector eigenfunctions, which share the eigenvalues κ^2, are generated by the scalar eigenfunctions $\psi_{\substack{e\\o}mn}(h)$ of the operator $-\nabla^2$ according to [4]

$$\boldsymbol{M}_{\substack{e\\o}mn}(h) = \nabla \times \hat{\boldsymbol{z}}\psi_{\substack{e\\o}mn}(h)$$
$$\boldsymbol{N}_{\substack{e\\o}mn}(h) = \nabla \times \boldsymbol{M}_{\substack{e\\o}mn}(h) \tag{12}$$

where ψ_{emn} and ψ_{omn} satisfy, respectively, the Neumann and Dirichlet boundary conditions at the waveguide walls.[2] For the rectangular waveguide, one easily finds

$$\psi_{\substack{e\\o}mn}(h) = \begin{cases} \cos\dfrac{m\pi x}{a} & \cos\dfrac{n\pi y}{b} \\ \sin\dfrac{m\pi x}{a} & \sin\dfrac{n\pi y}{b} \end{cases} e^{-jhz} \tag{13}$$

$$\kappa^2 = h^2 + k_c^2 \qquad k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{14}$$

The primed functions $\boldsymbol{M}'$ and $\boldsymbol{N}'$ in (11) depend on the source coordinates x', y', and z'. The meaning of the symbol δ_0 in (11) is as follows: $\delta_0 = 1$ when $m = 0$ or $n = 0$, and $\delta_0 = 0$ otherwise.

With (11) in mind, we now expand $\underline{\boldsymbol{G}}_A^s$ as [4]

$$\underline{\boldsymbol{G}}_A^s(\boldsymbol{r}|\boldsymbol{r}') = \int_{-\infty}^{\infty} dh \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{2-\delta_0}{\pi a b k_c^2} \cdot\left[a(h)\boldsymbol{M}_{emn}(h)\boldsymbol{M}'_{emn}(-h) + \frac{1}{\kappa^2}b(h)\boldsymbol{N}_{omn}(h)\boldsymbol{N}'_{omn}(-h)\right]. \tag{15}$$

To evaluate the coefficients $a(h)$ and $b(h)$ we substitute the expansions (11) and (15) into (5), introduce the operator ∇^2 under the integration and summation signs on the left side,[3] and make use of the relation $\nabla^2\boldsymbol{M} = -\kappa^2\boldsymbol{M}$ and a similar relation for the $\boldsymbol{N}$ functions. As a result, we find

$$a(h) = b(h) = \frac{1}{h^2 - k_g^2} \tag{16}$$

where

$$k_g = \begin{cases} \sqrt{k^2 - k_c^2}, & k_c < k \\ -j\sqrt{k_c^2 - k^2}, & k_c > k. \end{cases} \tag{17}$$

[2] For later convenience, we deliberately omit in (12) the inverse κ factor usually included in defining the $\boldsymbol{N}$ functions. Also, for notational simplicity, the dependence of κ on m, n, and h is not explicitly indicated throughout this paper. Similar remarks apply to k_c and k_g, introduced later.

[3] The conditions under which this change of the order of the operators is valid are discussed in [7].

We can now express (15) in the form

$$\underline{G}_A^s(\boldsymbol{r}|\boldsymbol{r}') = \frac{1}{k^2ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2-\delta_0}{\pi k_c^2} \cdot\left[\int_{-\infty}^{\infty}\frac{dh}{h^2-k_g^2}k^2\boldsymbol{M}_{emn}(h)\boldsymbol{M}'_{emn}(-h) + \int_{-\infty}^{\infty}\frac{dh}{h^2}\left(\frac{k_g^2}{h^2-k_g^2}+\frac{k_c^2}{h^2+k_c^2}\right) \cdot\boldsymbol{N}_{omn}(h)\cdot\boldsymbol{N}'_{omn}\cdot(-h)\right] \quad (18)$$

which clearly exhibits two sets of poles in the complex h plane, located at $\pm k_g$ and at $\pm jk_c$ ($h=0$ is a regular point of the integrand). The integrals in (18) can be easily evaluated by residue calculus and the result can be expressed as

$$\underline{G}_A^s(\boldsymbol{r}|\boldsymbol{r}') = \frac{1}{k^2}\left[\underline{G}_k(\boldsymbol{r}|\boldsymbol{r}') - \underline{G}_0(\boldsymbol{r}|\boldsymbol{r}')\right] \quad (19)$$

where

$$\underline{G}_k(\boldsymbol{r}|\boldsymbol{r}') = \frac{1}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2-\delta_0}{jk_gk_c^2} \cdot\left[k^2\boldsymbol{M}_{emn}(\pm k_g)\boldsymbol{M}'_{emn}(\mp k_g) + \boldsymbol{N}_{omn}(\pm k_g)\boldsymbol{N}'_{omn}(\mp k_g)\right], \quad z \gtrless z' \quad (20)$$

and

$$\underline{G}_0(\boldsymbol{r}|\boldsymbol{r}') = \frac{2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k_c^3}\boldsymbol{N}_{omn}(\mp jk_c)\boldsymbol{N}'_{omn}(\pm jk_c), \quad z \gtrless z'. \quad (21)$$

We observe that $\underline{G}_0$ is the static limit of $\underline{G}_k$, i.e.,

$$\underline{G}_0(\boldsymbol{r}|\boldsymbol{r}') = \lim_{k\to 0}\underline{G}_k(\boldsymbol{r}|\boldsymbol{r}'). \quad (22)$$

We can use $\underline{G}_A^s$ in place of $\underline{G}_E$ in (8) to find $\boldsymbol{E}^s$. To obtain $\boldsymbol{E}^l$, we must replace in that equation $\underline{G}_E$ by $\underline{G}_E^l$, which is easily derivable from the static scalar potential (see the Appendix). The complete electric Green's function can be obtained as a sum of $\underline{G}_A^s$ and $\underline{G}_E^l$. Hence, adding (19) and (37) we obtain

$$\underline{G}_E(\boldsymbol{r}|\boldsymbol{r}') = \frac{1}{k^2}\left[\underline{G}_k(\boldsymbol{r}|\boldsymbol{r}') - \hat{z}\hat{z}\delta(\boldsymbol{r}-\boldsymbol{r}')\right] \quad (23)$$

which is in agreement with the results derived by different methods by Tai [8] and Rahmat-Samii [9].[4]

Finally, we remark that the integral in (11) can also be evaluated, with the result

$$\underline{\delta}^s(\boldsymbol{r}-\boldsymbol{r}') = (\hat{x}\hat{x}+\hat{y}\hat{y})\delta(\boldsymbol{r}-\boldsymbol{r}') + \underline{G}_0(\boldsymbol{r}|\boldsymbol{r}'). \quad (24)$$

[4]There are misprints in this reference, as pointed out by Wang [10].

IV. Smythe's Approach

Smythe's approach [2] can be paraphrased as follows.[5] Since $\nabla\cdot\boldsymbol{A}^s = 0$, hence $\boldsymbol{A}^s$ is expressible in terms of two scalars, say ψ^h and ψ^e, Smythe postulates the solenoidal representation (cf. (2) in [2, sec. 13.00])

$$\boldsymbol{A}^s = \nabla\times\hat{z}\psi^h + \nabla\times\nabla\times\hat{z}\psi^e. \quad (25)$$

It is easily shown that (25) satisfies the homogeneous form of (2) provided that ψ^h and ψ^e are solutions of the homogeneous Helmholtz equation. Therefore, Smythe first seeks a solution of (2) outside the $z=z'$ source plane and postulates for ψ^h and ψ^e the forms[6]

$$\psi^h(x,y,z) = T^h(x,y)e^{\mp jk_g(z-z')}$$
$$\psi^e(x,y,z) = \mp T^e(x,y)e^{\mp jk_g(z-z')} \quad (26)$$

where, as before, the upper and lower signs pertain to $z>z'$ and $z<z'$, respectively. The separation of variables procedure results in the transverse eigenvalue problems

$$(\nabla_t^2 + k_c^2)T^{e,h} = 0 \qquad k_g^2 = k^2 - k_c^2$$
$$T^e = 0 \qquad \frac{\partial}{\partial n}T^h = 0 \qquad \text{on } S \quad (27)$$

where $\nabla_t^2 = \nabla^2 - (\partial^2/\partial z^2)$. One easily shows that ψ^h and ψ^e generate, respectively, fields TE and TM to the z direction, which was anticipated in the notation. In fact, these partial fields have the forms (cf. (14)–(17) in [2, sec. 13.00])

$$\boldsymbol{E}^h = j\omega\hat{z}\times\nabla_t T^h e^{-jk_g|z-z'|}$$
$$\boldsymbol{H}_t^h = \mp\frac{jk_g}{\mu}\nabla_t T^h e^{-jk_g|z-z'|}$$
$$H_z^h = \frac{k_c^2}{\mu}T^h e^{-jk_g|z-z'|} \quad (28)$$

and

$$\boldsymbol{E}_t^e = \omega k_g\nabla_t T^e e^{-jk_g|z-z'|}$$
$$E_z^e = \pm j\omega k_c^2 T^e e^{-jk_g|z-z'|}$$
$$\boldsymbol{H}^e = \pm\frac{k^2}{\mu}\hat{z}\times\nabla_t T^e e^{-jk_g|z-z'|} \quad (29)$$

where the subscript t denotes transverse components. The total field is a superposition of the TE and TM modes associated with the eigenvalues $k_c = k_{cmn}$, $m,n = 0,1,2\cdots$. The still arbitrary expansion coefficients are determined by enforcing the "jump" conditions at $z=z'$ [11]:

$$\hat{z}\times(\boldsymbol{H}^+ - \boldsymbol{H}^-) = \boldsymbol{J}_{st}$$
$$(\boldsymbol{E}^+ - \boldsymbol{E}^-)\times\hat{z} = -\frac{1}{j\omega\epsilon}\nabla_t\times\hat{z}J_{sz} \quad (30)$$

[5]Smythe assumes that $z>z'$, where z' is the source coordinate. In describing his approach, we extend it to also encompass the case $z<z'$.

[6]Observe that these forms do not satisfy the homogeneous Helmholtz equation when $z=z'$.

where J_{st} and J_{sz} are, respectively, the transverse and longitudial components of the surface current density, defined as

$$J_s(x, y) = \lim_{d \to 0} \int_{z'-d}^{z'+d} J(r)\, dz. \quad (31)$$

The superscripts + and − in (30) signify the limits of the corresponding quantities as z approaches z' from above and from below, respectively. Since the TE and TM modes are mutually orthogonal over the waveguide cross section, one can apply the jump conditions (30) separately to the TE and TM partial fields. This is the approach taken by Smythe, who uses conditions equivalent to (30) (cf. (5) and (11) in [2, sec. 13.03]).

We are now in a position to comment on the above procedure. First of all, we note that it is not correct to seek the solution of (2) *outside the source region* and to enforce the jump conditions at $z = z'$, since even for a current J localized at a point in space its solenoidal part J^s occupies the whole volume of the waveguide. The localized current assumption makes Smythe's approach tantamount to expanding the field outside the source in terms of the E and H waveguide modes, as can be seen by comparing (28) and (29) with [1, ch. 5, eqs. (6) and (11)]. As is now well known [12], this expansion gives the correct field everywhere in the waveguide, provided that E is augmented by the term $-\hat{z}J_z/(j\omega\epsilon)$. (That this term is needed is evident from the jump conditions (30), in which the z component of the electric current is represented in the second of these equations by an equivalent magnetic current, which is transverse to z. However, these sources are equivalent outside the current-carrying volume, but not inside, where the electric field due to the magnetic current must be modified as indicated above [13].) Hence, as a result of his procedure Smythe obtains the complete field due to the source J and not just its solenoidal part, as he intended.[7] He therefore errs when he states (cf. statements following (1) and (7) in [2, sec. 13.03]) that to the field thus obtained one must add the lamellar part, $E^l = -\nabla\Phi$, for that part is already contained in his solution. To recapitulate, Smythe in effect derives the Green's function $\underline{G}_E$ given in (23), and not—as he implies—the solenoidal Green's function $\underline{G}_A^s$ given in (19). The pitfalls of this approach of expanding the field in terms of the waveguide modes are easily avoided if one employs the eigenfunction expansion technique, as is demonstrated in Section III.

V. Discussion

The attractive feature of the Coulomb gauge is the explicit separation of the electric field into its lamellar and solenoidal constituents (cf. (3)). The lamellar part, which contains the dominant R^{-3} singularity, where R is the distance between the source and the observation points, is easy to determine, since the scalar potential Φ can be obtained by a simple differentiation of the corresponding static potential, as shown in the Appendix. (For a few simple geometries, this potential can be obtained in closed form by image theory.) The remaining part comprises the more manageable R^{-1} singularity and is, of course, solenoidal everywhere, including the source region. Hence, if the eigenfunction expansion technique is employed, E^s can be conveniently represented in terms of only the solenoidal M and N functions, and the lamellar L functions are indeed obviated [4].[8]

The price paid for these advantages of the Coulomb gauge is the added difficulty in solving for the vector potential if the approach is taken of expanding A^s (or E^s) in terms of the E and H waveguide modes [1], [12], [16]. This difficulty is due to the fact that (2), unlike the corresponding equation in the Lorentz gauge, involves the solenoidal part of J, which is usually a much more complicated function than J itself. For example, J associated with a point dipole has the simple form of the Dirac delta, whereas the corresponding J^s and J^l are not localized at a single point in space (cf. (24) and (38)). These difficulties in obtaining the Coulomb vector potential can perhaps be blamed for the subtle error in Smythe's book [2], which was written years before the nature of the field in the source region was fully explored.

As we concluded in the last section, the E and H modal expansion in effect employed by Smythe [2] leads to a vector potential that is not solenoidal in the source region. Denoting this potential by A', we therefore have $\nabla \cdot A' \neq 0$ for $r = r'$. There are at least two ways of correcting A' *post factum* to obtain the correct Coulomb potential, A^s. The first is to integrate A' against the solenoidal delta function in a manner indicated in (9). The second way is to put $A^s = A' + \nabla\psi$, where, by enforcing the condition $\nabla \cdot A^s = 0$, we find that the scalar function ψ can be found from $\nabla^2\psi = -\nabla \cdot A'$. Both of these methods appear to be more cumbersome than the eigenfunction expansion technique followed in Section III.

In summary, the advantages of the Coulomb gauge over the Lorentz gauge are to a considerable degree offset by the difficulties associated with its use. We also note in retrospect that the decomposition of the Green's function $\underline{G}_E$ into its solenoidal and lamellar parts can, if desired, be achieved without recourse to the Coulomb gauge by simply subtracting from and adding to $\underline{G}_E$ the static limit of $\underline{G}_k$ (cf. (19), (23), and (37)). This procedure is often followed to accelerate the convergence of the series that arise in problems involving sources in waveguides and cavities [3], [17]–[19].

To end on a more optimistic note, we observe from (3) that the Coulomb gauge leads to an alternative and possibly advantageous form of the so-called mixed-potential integral equation [20], which is amenable to the efficient numerical solution technique developed by Rao, Wilton, and Glisson [21]. The authors intend to pursue this promising aspect of the Coulomb gauge in a forthcoming paper.

[7] The multiplicative factor j in (6) and (9) of [2, sec. 13.03] is superfluous. Also, the sign of the z component in (6) should be changed to plus.

[8] The inclusion of the lamellar eignefunctions is required to obtain a complete representation of the *total* electric field in the source region. Their omission led to an error in [4] (cf. [7, 8, 14, 15]).

Appendix
Derivation of $\underline{G}_E^l$ for the Rectangular Waveguide

The dyadic Green's function for the lamellar part of the electric field can be assembled in a few simple steps beginning with the solution of the static problem

$$\nabla^2 G_\Phi(\mathbf{r}|\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{32}$$

where G_Φ must vanish at the waveguide walls and at infinity. Using Φ of (4) and G_Φ in Green's second identity [1] and referring to (3) and (8), one can show that

$$\underline{G}_E^l(\mathbf{r}|\mathbf{r}') = -\frac{1}{k^2}\nabla\nabla' G_\Phi(\mathbf{r}|\mathbf{r}'). \tag{33}$$

For the case of free space, this result reduces to that of Howard [22].

For the rectangular waveguide, we easily find [2][9]

$$G_\Phi(\mathbf{r}|\mathbf{r}') = \frac{2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k_c}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y'}{b} \cdot \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-k_c|z-z'|} \tag{34}$$

where k_c is defined in (14). The derivatives in (33) pose no difficulty, with the exception of the $\partial^2/\partial z\partial z'$ present in the $\hat{z}\hat{z}$ term of the dyadic, which contributes a Dirac delta, if one observes that [9], [23]

$$\frac{\partial^2}{\partial z \partial z'}e^{-k_c|z-z'|} = 2k_c\delta(z-z') - k_c^2 e^{-k_c|z-z'|}. \tag{35}$$

Using this result and the completeness relation [1]

$$\delta(x-x')\delta(y-y') = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x'}{a}\sin\frac{n\pi y'}{b} \cdot \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{36}$$

we easily show that

$$\underline{G}_E^l(\mathbf{r}|\mathbf{r}') = \frac{1}{k^2}\left[\underline{G}_0(\mathbf{r}|\mathbf{r}') - \hat{z}\hat{z}\delta(\mathbf{r}-\mathbf{r}')\right] \tag{37}$$

with $\underline{G}_0$ defined in (21).

Finally, upon comparing the last result with (10) we observe that

$$\underline{\delta}^l(\mathbf{r}-\mathbf{r}') = \hat{z}\hat{z}\delta(\mathbf{r}-\mathbf{r}') - \underline{G}_0(\mathbf{r}|\mathbf{r}'). \tag{38}$$

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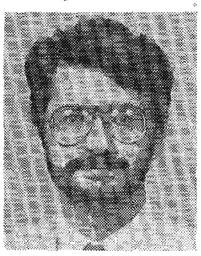

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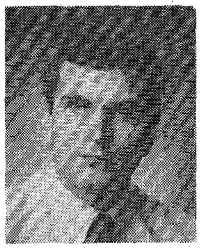

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